

MAXIMAL SUBALGEBRAS OF VECTOR FIELDS FOR EQUIVARIANT QUANTIZATIONS

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ABSTRACT. The elaboration of new quantization methods has recently developed the interest in the study of subalgebras of the Lie algebra of polynomial vector fields over a Euclidean space. In this framework, these subalgebras define maximal equivariance conditions that one can impose on a linear bijection between observables that are polynomial in the momenta and differential operators. Here, we determine which finite dimensional graded Lie subalgebras are maximal. In order to characterize these, we make use of results of Guillemin, Singer and Sternberg and Kobayashi and Nagano.

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I. INTRODUCTION

Our interest in the present study comes from recent works about new equivariant quantizations (Ref. 1, 2).

One can define *quantization maps* as linear bijections \mathcal{Q} from the space $Pol(T^*M)$ of functions on the cotangent bundle of a smooth manifold M , that are polynomial on the fibre, to a space $D_\lambda(M)$ of differential operators acting on tensor densities of weight λ over M .

It is known that a quantization map \mathcal{Q} cannot be equivariant with respect to all diffeomorphisms of M . From the infinitesimal point of view, this means that such a map does not commute with the action on these spaces of the Lie algebra $Vect(M)$ of vector fields over M . In other words, differential operators and polynomials are inequivalent modules of $Vect(M)$.

However, when M is endowed with an additional structure, some particular subalgebras of $Vect(M)$ naturally deserve consideration, because they are made up of infinitesimal transformations preserving the structure.

The authors of Ref. 1, 2 considered the case of infinitesimal projective or conformal transformations of M . In suitable charts, these can be realized in polynomial vector fields over a Euclidean space. For instance, if M is endowed with a projective structure (i.e. M is locally identified with a real projective space, say of dimension n) then in appropriate charts, the Lie algebra of infinitesimal projective transformations – isomorphic to $sl(n+1, \mathbb{R})$ – is generated by the vector fields

$$\frac{d}{dx^j}, x^j \frac{d}{dx^k}, x^j \sum_{l=1}^n x^l \frac{d}{dx^l}, \quad \forall j, k \leq n. \quad (1)$$

In this setting, those conformal and projective subalgebras share the property of being maximal in the algebra of polynomial vector fields: they are not contained in any larger proper subalgebra. The reader may refer to Ref. 2, 3 for proofs.

Now, it was proved in Ref. 1, 2 that one could construct a quantization map equivariant with respect to those subalgebras. This quantization is unique up to normalization.

In this framework, our concern in the present paper is to determine all finite dimensional graded subalgebras of polynomial vector fields over a given Euclidean space that are maximal.

Independently of quantization purposes, other maximality conditions have also been studied.

In Ref. 4, Kantor classified *irreducible transitively differential groups*. This notion gives rise, from the Lie algebraic point of view, to the class of finite dimensional graded Lie subalgebras of polynomial vector fields containing all constant vector fields. The author then seeks for irreducible (see Ref. 4, p. 1405 or below) subalgebras being maximal in this class.

Another more recent study is that of Post (Ref. 5). In this paper, a stronger grading requirement is imposed in order to define a class of finite

dimensional Lie algebras containing all constant vector fields. All maximal subalgebras of this class are then identified.

We point out two differences between the maximality conditions examined here and in these studies.

On the one hand, we impose fewer conditions on the subalgebras we consider, keeping only the requirements for a subalgebra to be graded and finite dimensional. On the other hand, the maximality property is not investigated inside a particular class of subalgebras, but in the general class of all subalgebras of polynomial vector fields.

Before giving our main result and a brief description of the tools we shall use, let us fix some notations.

Throughout this note, we assume that E is an n -dimensional vector space over \mathbb{K} , which is taken to be \mathbb{R} or \mathbb{C} . We shall deal with polynomial vector fields over E .

We denote by $Vect_*(E)$ the space of these vector fields, i.e. the space of polynomial maps from E to E . It is worth noticing that the vector fields considered when E is complex are thus holomorphic. Let $\{e_j, j = 1, \dots, n\}$ be a basis of E . Assume that $X, Y \in Vect_*(E)$ are written $X = \sum_{j=1}^n X^j e_j$ and $Y = \sum_{j=1}^n Y^j e_j$. We denote as usual by $[X, Y]$ the Lie bracket

$$\sum_{j,k} X^j \partial_j Y^k e_k - Y^j \partial_j X^k e_k,$$

where ∂_j represents the derivation $\frac{d}{dx^j}$ along the j -th axis. For the sake of convenience, we shall also use this notation to designate the j th vector of a basis of E . We denote by $ad(X)$ the map $Y \mapsto [X, Y]$.

We name *Euler vector field* the identity transformation of E . In a basis $\{\partial_j\}$, it reads

$$\mathcal{E}(x) = \sum x^j \partial_j.$$

It defines a natural grading on $Vect_*(E)$:

$$Vect_*(E) = \bigoplus_{p \geq -1} Vect_p(E)$$

where $Vect_p(E)$ denotes the space of eigenvectors of $ad(\mathcal{E})$ associated with the eigenvalue p , i. e. vector fields with homogeneous coefficients of degree $p + 1$.

We are interested in these *graded* subalgebras L :

$$L = \bigoplus_{-1 \leq p \leq r} L_p \quad \text{with } L_p = Vect_p(E) \cap L$$

which are maximal in $Vect_*(E)$. As mentioned above, the notion of maximality has been used in various senses. Therefore, it is worth emphasizing the following definition.

Definition 1. A subalgebra L of $\text{Vect}_*(E)$ is *maximal* if

$$L \subset L' \Rightarrow L' = L \quad \text{or } L' = \text{Vect}_*(E)$$

whenever L' is a subalgebra of $\text{Vect}_*(E)$.

II. MAIN RESULT

Definition 2 (see for instance Ref. 6, p. 682). A graded subalgebra $L = L_{-1} \oplus \cdots \oplus L_k$ of $\text{Vect}_*(E)$ is said to be *irreducible* if the representation $(L_{-1}, \text{ad}|_{L_0})$ is irreducible.

Theorem 1. Let $L = L_{-1} \oplus \cdots \oplus L_k$ be a graded subalgebra of $\text{Vect}_*(E)$. Then L is maximal if and only if

1. $L_{-1} = \text{Vect}_{-1}(E)$,
2. L is irreducible,
3. $L_1 \neq 0$,
4. when $\mathbb{K} = \mathbb{R}$, the representation $(L_{-1}, \text{ad}|_{L_0})$ admits no complex structure.

The text below is organized as follows. In Section III, we prove the necessity of the first three conditions above. Then, in Section IV, we consider polynomial vector fields from a slightly modified point of view, in order to prove, in Section V, the fourth condition given above. We expose in Section VI how the graded maximal subalgebras relate to the irreducible filtered Lie algebras of finite type, which were classified in Ref. 7. Using the classification of all irreducible infinite dimensional subalgebras of polynomial vector fields (see for instance Ref. 8, 9, 10, 6, and references therein), we show in Section VII that all these algebras give rise to a canonical graded maximal subalgebra of polynomial vector fields.

III. CONSTANT VECTOR FIELDS AND IRREDUCIBILITY

Lemma 1. Let L be a maximal subalgebra of $\text{Vect}_*(E)$. Then $\mathcal{E} \in L$ if and only if L is graded.

Proof. The sufficiency of the condition follows from the fact that

$$\mathbb{K}\mathcal{E} + L$$

is a Lie subalgebra when L is graded. In order to check the necessity of the condition, notice that

$$\text{ad}(\mathcal{E})^k L \subset L, \quad \forall k \in \mathbb{N},$$

gives a Vandermonde system allowing to compute the homogeneous components of a vector field $X \in L$. \square

This proof is similar to the proof by Koecher (see Ref. 11, p. 354) that any ideal of $\text{Vect}_*(E)$ is graded. We therefore state the following remark.

Remark 1. If L is a subalgebra of $\text{Vect}_*(E)$ that contains \mathcal{E} , then any ideal of L is graded.

Lemma 2. *Let L_{-1} , L_0 and L_+ be vector subspaces of $\text{Vect}_{-1}(E)$, $\text{Vect}_0(E)$ and $\bigoplus_{i \geq 1} \text{Vect}_i(E)$ respectively, such that*

1. $L_{-1} \oplus L_0$ is a Lie subalgebra
2. $[L_{-1}, L_+] \subset L_0 \oplus L_+$, and $[L_0, L_+] \subset L_+$.

Set $c^0(L_+) = L_+$ and $c^{k+1}(L_+) = [L_+, c^k(L_+)]$, ($k \in \mathbb{N}$).

Then the smallest Lie subalgebra containing

$$L_{-1} \oplus L_0 \oplus L_+$$

is

$$L_{-1} \oplus L_0 \oplus \sum_{k \in \mathbb{N}} c^k(L_+).$$

In particular, if $L_+ \subset \text{Vect}_1(E)$, the latter subalgebra is graded.

Proof. Using Jacobi identity, we check that $[L_0, c^k(L_+)] \subset c^k(L_+)$ and consequently $[L_{-1}, c^k(L_+)] \subset \sum_{i=0}^k c^i(L_+)$ by induction on $k \geq 1$. By definition, $[c^0(L_+), c^k(L_+)] = c^{k+1}(L_+)$ for all $k \in \mathbb{N}$. Then, we check, by induction on $j \geq 0$, that $[c^j(L_+), c^k(L_+)] \subset c^{j+k+1}(L_+)$.

Therefore, $L_{-1} \oplus L_0 \oplus \sum_{k \in \mathbb{N}} c^k(L_+)$ is a Lie subalgebra. It is trivially the smallest one to contain the subspaces L_{-1} , L_0 and L_+ . \square

Definition 3. Let F be a vector subspace of $\text{Vect}_{-1}(E)$. We set

$$\mathcal{N}^i(F) = \{X \in \text{Vect}_i(E) : \text{ad}(F)^{i+1}X \subset F\}$$

and

$$\mathcal{N}(F) = \bigoplus_{i \geq -1} \mathcal{N}^i(F).$$

Notice that $\mathcal{N}^{-1}(F) = F$ and that $\mathcal{N}^0(F)$ is the intersection of the normalizer of F and the subspace of linear vector fields.

Proposition 2. *Let $L = \bigoplus_{i \geq -1} L_i$ be a graded subalgebra of $\text{Vect}_*(E)$. Then $\mathcal{N}(L_{-1})$ is an infinite dimensional graded subalgebra containing L . Moreover, $\mathcal{N}(L_{-1}) = \text{Vect}_*(E)$ if and only if $L_{-1} = \text{Vect}_{-1}(E)$.*

Proof. It is obvious that $[\mathcal{N}^i(L_{-1}), \mathcal{N}^j(L_{-1})] \subset \mathcal{N}^{i+j}(L_{-1})$. Furthermore, if $L_{-1} = 0$ or $L_{-1} = \text{Vect}_{-1}(E)$,

$$\mathcal{N}(L_{-1}) = L_{-1} \oplus \bigoplus_{i \geq 0} \text{Vect}_i(E).$$

Now, if $h \in L_{-1}$, then, for every polynomial function $p : E \rightarrow \mathbb{K}$, the field $x \mapsto p(x)h$ belongs to $\mathcal{N}(L_{-1})$. \square

Corollary 3. *Let L be a finite dimensional graded maximal subalgebra of $\text{Vect}_*(E)$. Then $\text{Vect}_{-1}(E) \subset L$.*

Corollary 4. *Let L be a finite dimensional graded maximal subalgebra of $\text{Vect}_*(E)$. Then $L_1 \neq 0$.*

Proof. Notice that L cannot be made only of constant and linear vector fields. Indeed, it would then be included in the maximal subalgebra (1) presented in the introduction, for instance. Therefore, $L_k \neq 0$ for some $k > 0$. The conclusion follows from Corollary 3. \square

Proposition 5. *Let L be a finite dimensional graded maximal subalgebra of $\text{Vect}_*(E)$. Then*

$$(L_{-1} = \text{Vect}_{-1}(E), \text{ad}_{|L_0})$$

is an irreducible representation of L_0 . It follows that any non trivial ideal of L contains every constant vector field.

Proof of Proposition 5. Let $F \neq \{0\}$ be a stable subspace of L_{-1} under the action of L_0 .

The space

$$L_{-1} \oplus \mathcal{N}^0(F) \oplus \bigoplus_{i \geq 1} \{X \in \text{Vect}_i(E) : \text{ad}(L_{-1})^i X \subset \mathcal{N}^0(F)\}$$

satisfies the hypotheses of Lemma 2. Its algebraic closure is an infinite dimensional proper subalgebra containing L properly, hence a contradiction.

Let now I be a non trivial ideal of L . It contains at least one constant vector field since $[\text{Vect}_{-1}(E), I] \subset I$. It contains all of them since $I \cap L_{-1}$ is a stable subspace of L_{-1} . \square

IV. A CONVENIENT MODEL FOR POLYNOMIAL VECTOR FIELDS

It will be useful to consider the spaces of multilinear symmetric mappings from $E \times \cdots \times E$ to E instead of those of homogeneous polynomial vector fields. We shall write

$$\mathcal{T}_i(E) = S^{i+1}E^* \otimes E, \quad \text{and } \mathcal{T}_*(E) = \bigoplus_{i \geq -1} \mathcal{T}_i(E).$$

To turn $\mathcal{T}_*(E)$ into a Lie algebra, we define as in Ref. 6 the following bracket operation. If $t \in \mathcal{T}_p(E)$ and $t' \in \mathcal{T}_q(E)$ then $[t, t'] \in \mathcal{T}_{p+q}(E)$ and

$$\begin{aligned} [t, t'](x_0, x_1, \dots, x_{p+q}) = & \\ & \frac{1}{p!(q+1)!} \sum_j t(t'(x_{j_0}, x_{j_1}, \dots, x_{j_q}), x_{j_{q+1}}, \dots, x_{j_{p+q}}) \\ & - \frac{1}{(p+1)!q!} \sum_k t'(t(x_{k_0}, x_{k_1}, \dots, x_{k_p}), x_{k_{p+1}}, \dots, x_{k_{p+q}}) \end{aligned}$$

where both j and k run over all possible permutations of the $p+q+1$ first natural numbers.

Proposition 6. *The map $T : \mathcal{T}_*(E) \rightarrow \text{Vect}_*(E)$ defined by*

$$T(M) : x \in E \mapsto -\frac{1}{(p+1)!}M(x, \dots, x), \quad \forall M \in \mathcal{T}_p(E)$$

is an isomorphism of Lie algebras.

V. ABSENCE OF COMPLEX STRUCTURE

We now assume $\mathbb{K} = \mathbb{R}$ and prove, in Lemma 3, the fourth condition of maximality of our main result.

Let E be a real vector space of even dimension and J a complex structure of E , i.e. an endomorphism of E such that $J^2 = -\text{id}$. We denote by E_J the complex vector space defined by E with the structure of \mathbb{C} -module defined by

$$(a + ib)e := ae + bJe, \quad \forall a, b \in \mathbb{R}, \forall e \in E.$$

Define

$$\begin{aligned} \mathcal{T}_p^J(E) = \\ \{M \in \mathcal{T}_p(E) \mid J(M(x_0, \dots, x_p)) = M(Jx_0, x_1, \dots, x_p), \forall x_0, \dots, x_p \in E\} \end{aligned}$$

for all $p \geq -1$. Then the subalgebra $\mathcal{T}_*^J(E) = \bigoplus_{i \geq -1} \mathcal{T}_i^J(E)$ of $\mathcal{T}_*(E)$ is isomorphic to $\mathcal{T}_*(E_J)$ as a real Lie algebra. Indeed, the condition defining $\mathcal{T}_*^J(E)$ means that an application $M \in \mathcal{T}_*(E)$ is \mathbb{C} -multilinear on E_J .

Lemma 3. *Let E be a real vector space and $L = \mathcal{T}_{-1}(E) \oplus \bigoplus_{j=0}^k L_j$ a graded subalgebra of $\mathcal{T}_*(E)$. Assume that J is a complex structure of $(L_{-1}, \text{ad}|_{L_0})$, i.e.*

$$\begin{aligned} [x_0, Jx_{-1}] &= J[x_0, x_{-1}], \quad \forall x_0 \in L_0, \forall x_{-1} \in L_{-1}, \\ \text{and } J^2 &= -\text{id}. \end{aligned}$$

Then

$$L \subset \mathcal{T}_*^J(L_{-1}) \subset \mathcal{T}_*(L_{-1})$$

where both inclusions are strict.

Proof. Indeed, $L_{-1} = \mathcal{T}_{-1}^J(E) = \mathcal{T}_{-1}(E)$. The requirement for J to intertwine the action of L_0 on L_{-1} precisely means that $L_0 \subset \mathcal{T}_0^J(E)$. If $L_{k-1} \subset \mathcal{T}_{k-1}^J(E)$ and $M \in L_k$, the equalities

$$\begin{aligned} J \circ M(x_0, \dots, x_k) &= J([M, x_1](x_0, x_2, \dots, x_k)) \\ &= M(x_1, Jx_0, x_2, \dots, x_k) \\ &= M(Jx_0, x_1, \dots, x_k) \end{aligned}$$

show that $M \in \mathcal{T}_k^J(E)$.

The inclusions are strict because the dimension of $\mathcal{T}_*^J(L_{-1})$ is infinite and because the dimension of $\mathcal{T}_p^J(L_{-1})$, for all $p \geq 0$, is strictly less than that of $\mathcal{T}_p(L_{-1})$. \square

This lemma generalizes the construction used in Ref. 3 to show that a subalgebra of infinitesimal conformal transformations, isomorphic to $so(3, 1, \mathbb{R})$, is not maximal in $Vect_*(\mathbb{R}^2)$.

VI. IRREDUCIBLE FILTERED ALGEBRAS OF FINITE TYPE

Let $L = \bigoplus_{j=-1}^k L_j$ be a graded maximal subalgebra of polynomial vector fields. In the last section, we have shown that L possesses interesting properties. It actually belongs to a broader class of Lie algebras studied in (Ref. 7, Theorem 1, p. 875).

This theorem describes the structure of some filtered finite dimensional Lie algebras together with a group of automorphisms.

We shall only associate the trivial group $\{\text{id}\}$ to such an algebra. Furthermore, the reader may find worth noticing that the algebras we consider carry the filtration which is naturally associated to their grading and that the other hypotheses of the theorem are satisfied in view of the first three conditions required in our main result for a subalgebra to be maximal.

For the sake of simplicity, we shall name algebras described by this theorem *Irreducible filtered algebras of finite type*, as it was done in Ref. 6, or simply write *IFFT-algebras*.

As a consequence of the mentioned result, we know that L is simple and is of order two, i.e. $L = L_{-1} \oplus L_0 \oplus L_1$. Moreover, there exists a unique element $e \in L$ such that L_p is the eigenspace of $ad(e)$ associated to the eigenvalue p . This element is in the center of L_0 . We shall name it the *Euler element* of L . Finally, L_{-1} and L_1 are dual to each other as modules of L_0 with respect to the Killing form of L .

On the one hand, Kobayashi and Nagano gave a list of the admissible algebras and detailed in each case the associated grading. The pairs (L, e) where L is a real IFFT-algebra and e its Euler element are classified in Ref. 7, pp. 892–895. On the other hand, to any graded algebra $L = \bigoplus_{k \geq -1} L_k$, they associated in a natural way a graded subalgebra of $\mathcal{T}_*(L_{-1})$ (see Ref. 6, p. 683). The reader may compare this construction with that of Gradl (Ref. 12). In the case of $L = L_{-1} \oplus L_0 \oplus L_1$, this is done by the following monomorphism $\phi : L \rightarrow \mathcal{T}_*(L_{-1})$:

$$\begin{cases} \phi|_{L_{-1}} &= \text{id} \\ \phi|_{L_0} &= ad|_{L_0} \\ \phi(M) &= (x, y) \mapsto [[M, x], y], \quad \forall M \in L_1, \forall x, y \in L_{-1}. \end{cases}$$

Notice that this is the only way to proceed provided the value of ϕ on L_{-1} is set to id.

VII. IFFT-ALGEBRAS ARE MAXIMAL

In this section, we prove the sufficiency of the conditions given in our main result for a subalgebra of polynomial vector fields to be maximal.

We first assume that E is a complex vector space and $L = L_{-1} \oplus L_0 \oplus L_1$ an irreducible graded subalgebra of $\mathcal{T}_*(E)$ such that $L_{-1} = \mathcal{T}_{-1}(E)$ and $L_1 \neq 0$. Then we shall show how the proof adapts to the real case.

Let L' be a subalgebra of $\mathcal{T}_*(E)$ such that $L' \supset L$. Then L' is graded and irreducible, since L is.

If L' is finite dimensional, one sees, by using the description of the IFFT-algebras (see Section VI), that $L'_1 = L_1$, that L' is simple and eventually that $L = L'$, since $[L'_{-1}, L'_1] = L_0$.

Therefore, if L' contains properly L , then it must be infinite dimensional. It possesses two additional properties, consequences of the following result.

Proposition 7 (Ref. 6, p. 688). *Let $\bigoplus_{p \geq -1} G_p$ be an irreducible graded Lie algebra of infinite type or finite type of order ≥ 2 over a field of characteristic 0. Then G_0 is reductive and $[G_{-1}, G_1]$ contains the semi-simple part of G_0 .*

- L'_0 is reductive and has a non trivial center (the multiples of the identity transformation of L_{-1}).
- L' is still simple. Indeed, if I is an ideal of L' then $I \supset L$ (see Proposition 5), which implies that I contains the multiples of the identity transformation of L_{-1} and in turn that $I \supset L'_j$ for all $j \neq 0$. Since $[L'_{-1}, L'_1]$ contains the semi-simple part of L'_0 , the conclusion follows.

In order to prove the maximality of L , the remaining point is to ensure that $L' = \mathcal{T}_*(E)$. The key result is due to Cartan. We refer the reader to the works of Guillemin, Quillen, Singer, Sternberg, Kobayashi and Nagano (Ref. 8, 9, 10, 6).

This result states that the only irreducible infinite dimensional graded subalgebras of $\text{Vect}_*(E)$ are

1. $\text{Vect}_*(E)$ itself,
2. the divergence-free vector fields,
3. the Hamiltonian vector fields with respect to a symplectic form given on E , provided E is even dimensional,
4. the last two subalgebras supplemented with the multiples of the Euler vector field.

But the subalgebras described in (4) are not simple, and those in (2) and (3) have a simple linear part.

Hence the proof.

Now, when E is a real vector space and L and L' as above, one proceeds in the same way to prove the simplicity of L' , noticing that both L_0 and L'_0 still have a one dimensional center.

Indeed, if x_0 is central in one of these two subalgebras, then $\text{ad}(x_0)$ intertwines the action of L_0 on L_{-1} . Since the representation $(L_{-1}, \text{ad}|_{L_0})$ admits no complex structure, Schur's lemma ensures that $\text{ad}(x_0)|_{L_{-1}}$ is a multiple of the identity transformation of L_{-1} . Therefore, $\dim Z(L_0) = 1$.

The description of irreducible infinite dimensional graded subalgebras of $\text{Vect}_*(E)$ is essentially due to Matsushima. It can be found in Ref. 6, 13.

Two cases arise whether $L'_0 \otimes \mathbb{C}$ acts irreducibly on $E \otimes \mathbb{C}$ or not. In the first case, L' should be one of the real analogues of the Cartan algebras listed above. But in the second, E admits a complex structure as a L_0 module, which contradicts the hypotheses.

Theorem 1 is proved.

In order to complete our search for maximal subalgebras of polynomial vector fields over a given *real* vector space, we need to be able to identify in the tables given in Ref. 7, pp. 892–895, the algebras such that the representation $(L_{-1}, ad|_{L_0})$ admits a complex structure.

Proposition 8. *Let L_{-1} be a real vector space and $L = L_{-1} \oplus L_0 \oplus L_1$ an IFFT-algebra. Then $(L_{-1}, ad|_{L_0})$ admits a complex structure if and only if the algebra L admits a complex structure.*

Proof. The sufficiency of the condition is obvious. Notice that a complex structure on L stabilizes the eigenspaces of e .

Let J_{-1} be a complex structure on $(L_{-1}, ad|_{L_0})$. Let $J_1 : L_1 \rightarrow L_1$ be the adjoint of J_{-1} with respect to the Killing form β of L , i.e.

$$\beta(J_1 x_1, x_{-1}) = \beta(x_1, J_{-1} x_{-1}), \quad \forall x_{-1} \in L_{-1}, \forall x_1 \in L_1.$$

The so defined J_1 intertwines the action of L_0 on L_1 . Moreover,

$$[J_1 x_1, x_{-1}] = [x_1, J_{-1} x_{-1}], \quad \forall x_{-1} \in L_{-1}, \forall x_1 \in L_1.$$

Indeed, for all $x_{-1}, y_{-1} \in L_{-1}$ and $x_1, y_1 \in L_1$,

$$\begin{aligned} \beta([x_1, J_{-1} x_{-1}], y_{-1}, y_1) &= \beta(J_{-1} x_{-1}, [x_1, [y_1, y_{-1}]]) \\ &= \beta(x_{-1}, [J_1 x_1, [y_1, y_{-1}]]) \\ &= \beta([J_1 x_1, x_{-1}], y_{-1}, y_1). \end{aligned}$$

Define

$$J_0 : L_0 \rightarrow \mathcal{T}_0(L_{-1}) : A \mapsto A \circ J_{-1}.$$

This map is actually valued in L_0 since

$$\begin{aligned} (J_0[x_1, x_{-1}])y_{-1} &= [[x_1, x_{-1}], J_{-1}y_{-1}] \\ &= [[x_1, J_{-1}y_{-1}], x_{-1}] \\ &= [[J_1x_1, y_{-1}], x_{-1}] \\ &= [J_1x_1, x_{-1}]y_{-1} \end{aligned}$$

for all $y_{-1} \in L_{-1}$.

The map $J : L \rightarrow L$ defined by its restrictions J_i to L_i ($i = -1, \dots, 1$) is then a complex structure of L as a Lie algebra. \square

The statement “IFFT-algebras are maximal” should be taken in the following sense. In the tables given in Ref. 7, one can distinguish complex algebras from real ones admitting no complex structure. The latter give rise to maximal subalgebras of the real algebra $\mathcal{T}_*(L_{-1})$. One may consider the former as Lie subalgebras of the real Lie algebra $\mathcal{T}_*(L_{-1})$, in which case

Lemma 3 shows that they are not maximal. They are maximal when regarded in their natural position of complex subalgebras of the complex Lie algebra $\mathcal{T}_*(L_{-1})$.

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